Newton methods for structured nonsmooth optimization

proximal identification and fast local convergence

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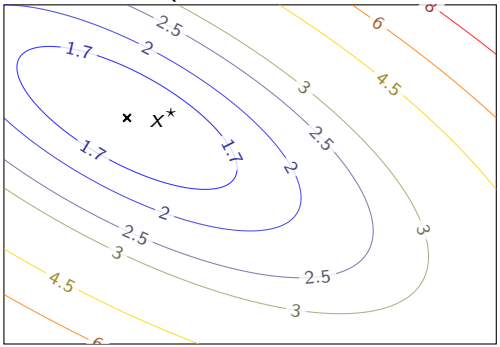
Reviewer	Claudia Sagastizábal
Examiner	Jean-Charles Gilbert
Examiner	Mathurin Massias
Advisor	Franck Iutzeler
Advisor	Jérôme Malick
Guest	Claude Lemaréchal

What this PhD is about

Optimization of functions: $\min_x F(x)$

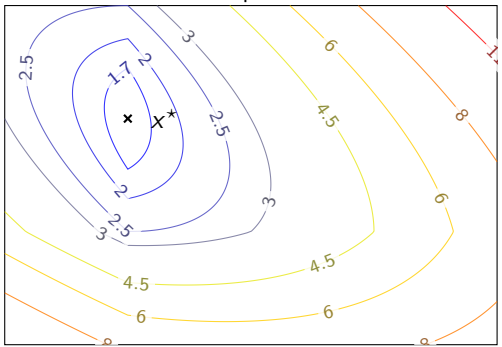
Quadratic function



$F(x) = \|Ax - b\|_2^2$

Smooth

Lasso problem



$F(x) = \|Ax - b\|_2^2 + \lambda\|x\|_1$

Nonsmooth

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Optimization of nonsmooth functions: $\min_x F(x)$

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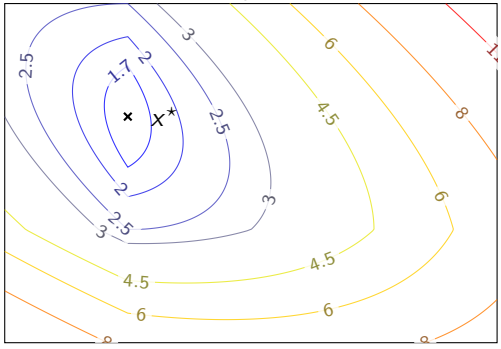

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Optimization of **structured** nonsmooth functions: $\min_x F(x)$

▹ nonsmooth points are "well-behaved"

on x & y axes

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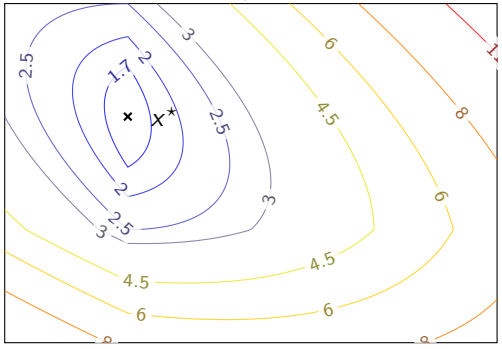

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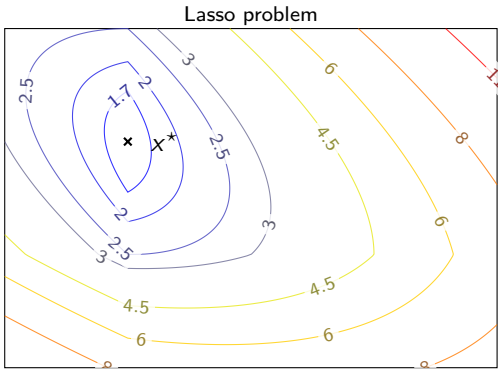

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Ok for lasso.
What about **structure in general nonsmooth functions**?

Sources of nonsmoothness 1

▷ **Implicit** nonsmoothness

$$F(x) = \sup_{u \in U} h(x, u)$$

Examples

- Robust optimization ◇ Ben-Tal *et al* '09
- Decomposition methods *e.g.*, Lagrangian relaxation, Benders decomposition ◇ BGLS '06
- Risk-averse optimization... advertising my team:
  - superquantile optim. ◇ Laguel Malick Harchaoui '22
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▷ *Full* **first-order** description of F near x *e.g.*, $\partial F(x)$?

This would require the *full* set $\arg\max_{u \in U} h(x, u)$.

Sources of nonsmoothness 2

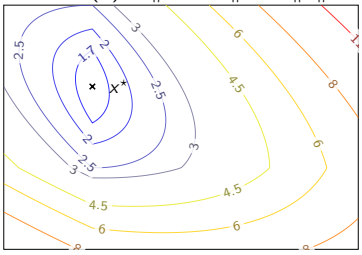
▷ **Chosen** nonsmoothness to attract minimizers

$$F(x) = f(x) + g(x), \quad \text{with } f \text{ smooth, } g \text{ nonsmooth}$$

Examples inverse problems, sparse regression *e.g.* lasso

◇ Scherzer *et al* '09, Vaiter *et al* '15

$\min_x F(x) = \|Ax - b\|^2 + \lambda\|x\|_1$



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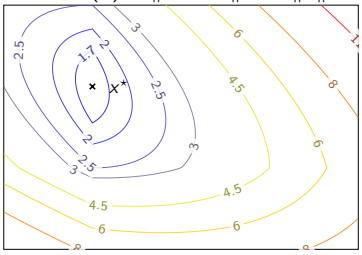

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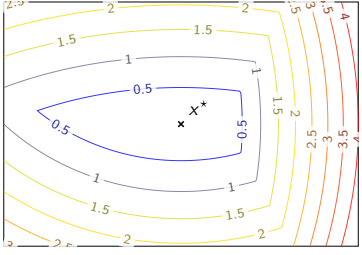

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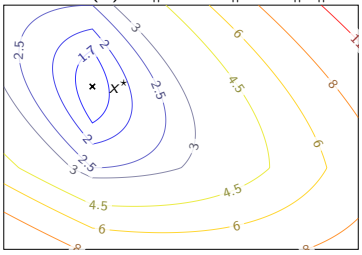

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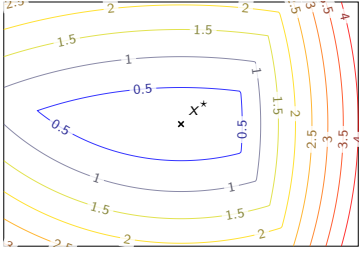

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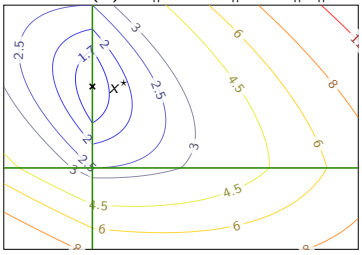

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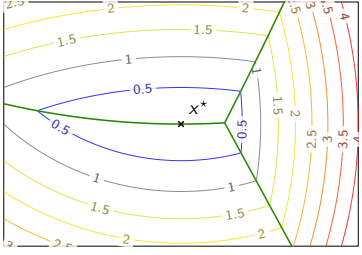

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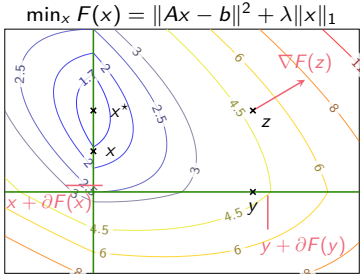

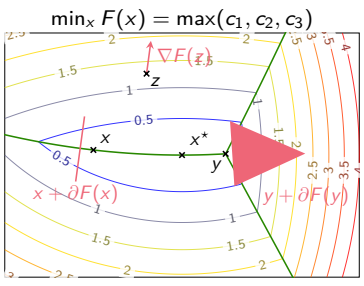

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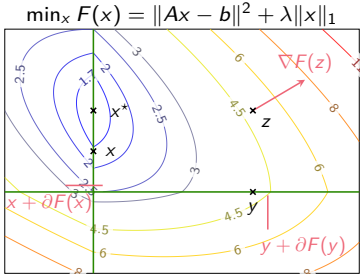
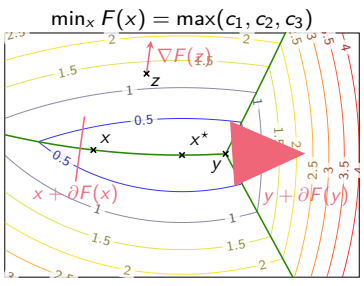
  Here **nonsmoothness is explicit**. Focus of this talk

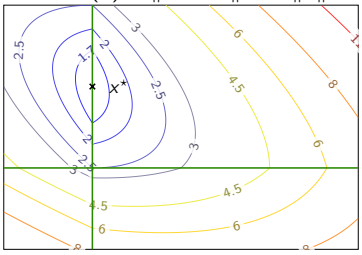




Structure in nonsmoothness

In most target applications, we **observe** that:

$\min_x F(x) = \|Ax - b\|^2 + \lambda\|x\|_1$



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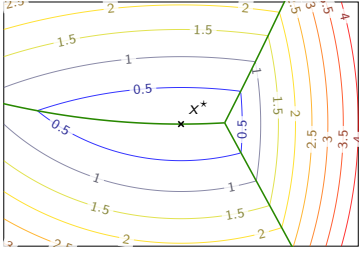

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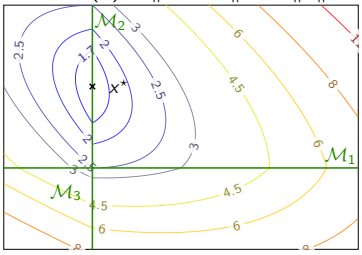

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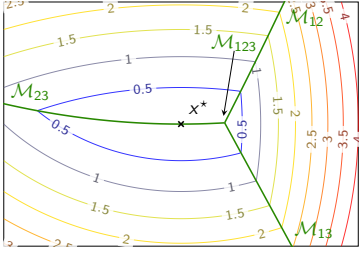

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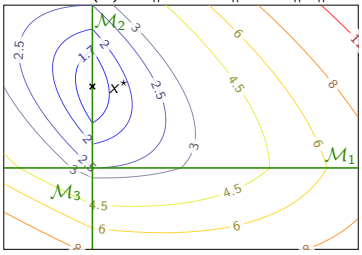

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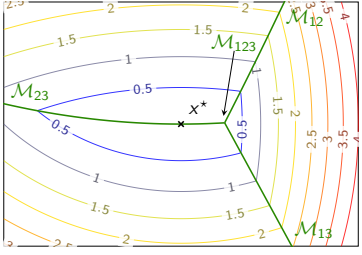

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These are **structure manifolds**.

◇ Lewis '02, Fadili Malick Peyré '18, Davis Drusviatsky '19

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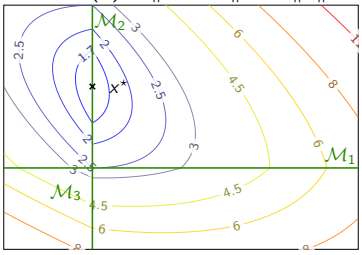

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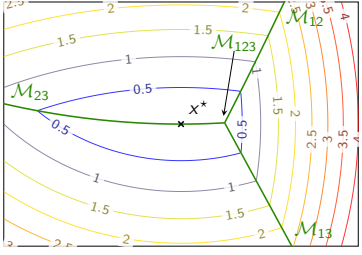

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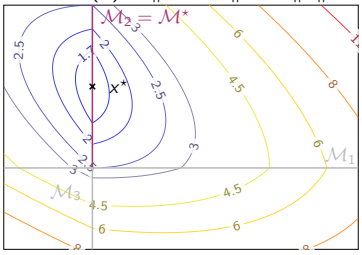
If $x^\star$ is nonsmooth for F, there is an

optimal manifold $\mathcal{M}^\star \ni x^\star$

→ This PhD explores the following question

Can structure help optimization?

$\min_x F(x) = \|Ax - b\|^2 + \lambda\|x\|_1$



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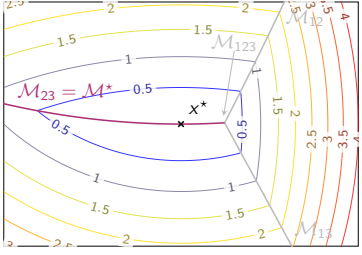
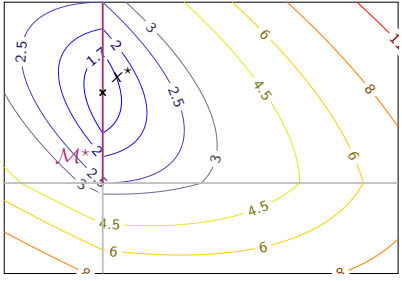


1. Detecting structure



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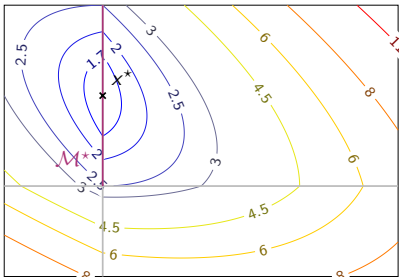
- **Implicit** nonsmoothness: $x \mapsto F(x)$, $v \in \partial F(x)$
  - *nonsmooth BFGS* ◇ Lewis Overton '13
  - gradient sampling ◇ Lewis *et al* '05
  - bundle methods ◇ HULL '93



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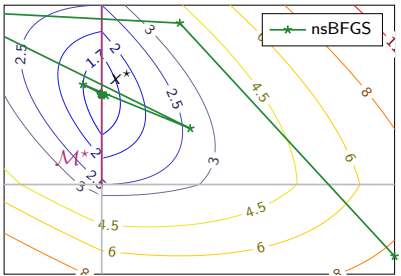

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→ **no detection** of $\mathcal{M}^\star$, suffered nonsmoothness

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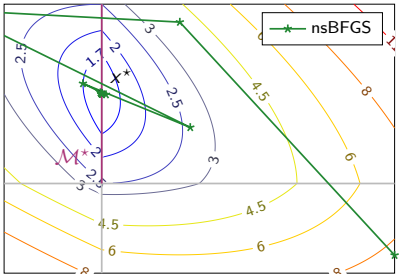
- **Explicit** nonsmoothness: $\mathbf{prox}_{\gamma g}$
  - *splitting methods* ◇ Combettes Pesquet, '11
  - prox-linear methods ◇ Lewis Wright, '16

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Implicit first-order step

Closed-form for simple nonsmooth functions



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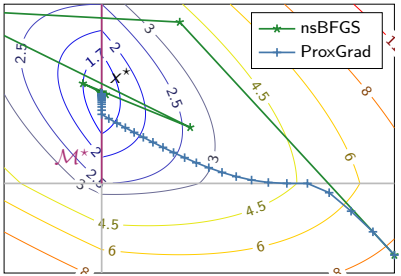

▹ nsBFGS: $x_k \notin \mathcal{M}^\star$, oscillations
→ **no detection** of $\mathcal{M}^\star$, suffered nonsmoothness

▹ ProxGrad: after some time $x_k \in \mathcal{M}^\star$
→ **detection** of $\mathcal{M}^\star$, but **no exploitation**

2. Exploiting structure

▷ With knowledge of $\mathcal{M}^\star$, nonsmooth min. turns into smooth constrained opt.

$$\min_{x \in \mathbb{R}^n} F(x) \qquad \overset{\text{with } \mathcal{M}^\star}{\longrightarrow} \qquad \min_{x \in \mathcal{M}^\star} F|_{\mathcal{M}^\star}(x)$$

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- Tune splitting method parameters $\to$ linear rate ◇ Liang Fadili Peyré '17
- $\mathcal{VU}$-algorithm $\to$ superlinear steps ◇ Mifflin Sagastizábal '05
  Approximate Newton on smooth directions, approximate prox

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Problem: $\mathcal{M}^\star$ is not known, only approximated numerically
Except in specific cases ◇ Daniilidis Sagastizábal Solodov '09

This thesis: detecting & exploiting structure

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- Smooth reduced problem

$$\min_{x \in \mathbb{R}^n} F(x) \qquad \overset{\text{with } \mathcal{M}^\star}{\longrightarrow} \qquad \min_{x \in \mathcal{M}^\star} F|_{\mathcal{M}^\star}(x)$$

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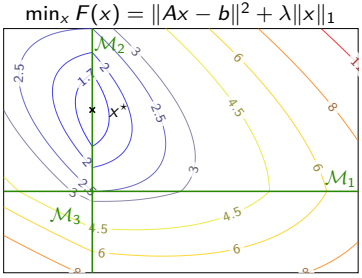
▷ We must use structure **adaptively**, which raises two difficulties:

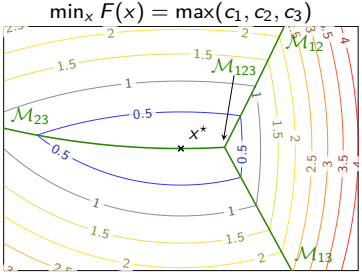
- How to *provably* **detect** relevant structure near a point x_k? → $\mathbf{prox}_{\gamma g}$
- How to **exploit** the detected structure $\mathcal{M}_k$? → Newton method

Outline

1. Additive nonsmoothness: $\min_x f(x) + g(x)$
   Iterates naturally live on $\mathcal{M}$, feasible optim. methods

2. Composite nonsmoothness: $\min_x g \circ c(x)$
   Iterates don't live on $\mathcal{M}$, infeasible optim. methods





Outline

Additive problems

$$\text{Find } x^\star \in \arg\min_{x \in \mathbb{R}^n} F(x) = \underbrace{f(x)}_{\text{smooth}} + \underbrace{g(x)}_{\text{non smooth}}$$

Example:

In inverse problems / learning, g is **chosen** to encode **prior knowledge**:

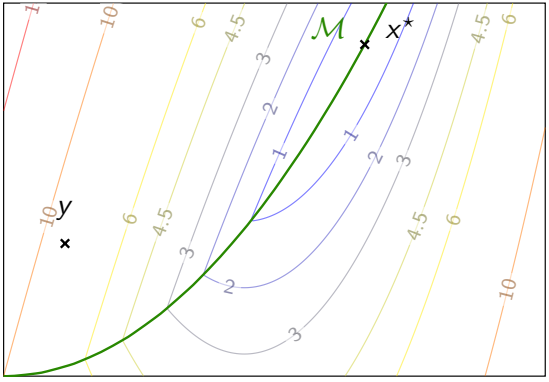
- $x^\star$ sparse $\longrightarrow g(x) \propto \|x\|_1 = \sum_{i=1}^n |x_i|$
- $X^\star$ low-rank $\longrightarrow g(X) \propto \|X\|_* = \sum_{i=1}^{\mathrm{rank}(X)} \sigma_i(X)$

◇ Candès Romberg Tao '06; Candès Recht '13; Vaiter Peyré Fadili '15

Back on the proximal gradient

- Proximal gradient algorithm: iterate

$$x = \mathbf{prox}_{\gamma g}(y - \gamma \nabla f(y))$$



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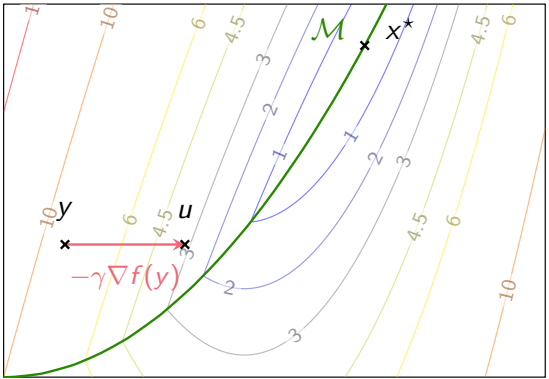

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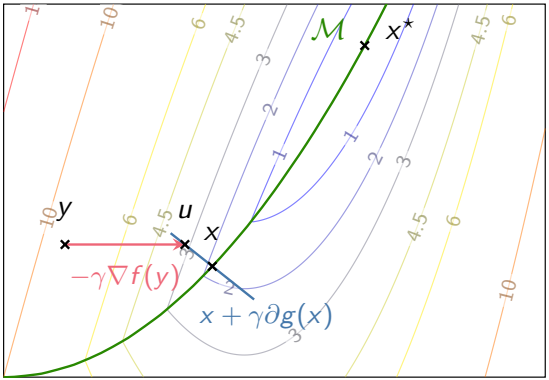
$$x = \mathbf{prox}_{\gamma g}(u)$$
$$\Leftrightarrow \quad u \in x + \gamma \partial g(x)$$

$\rightarrow$ $\mathbf{prox}_{\gamma g}$ can send points to $\mathcal{M}$



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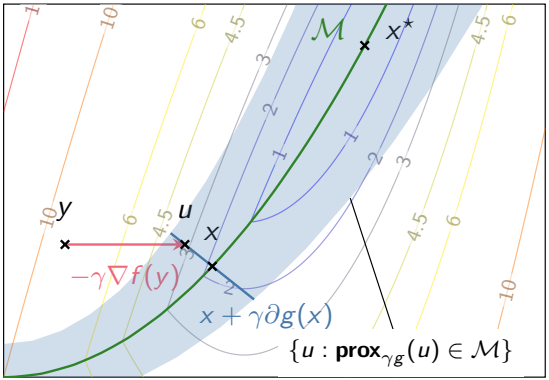

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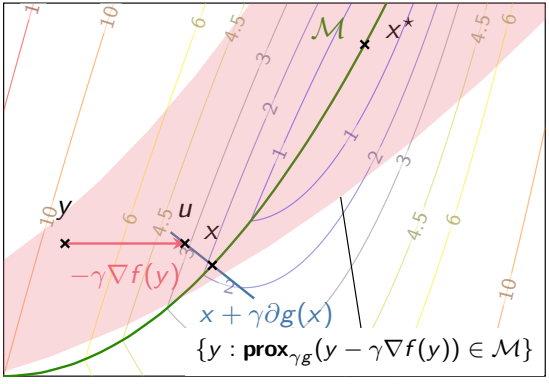

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Proximal gradient: a structure detector

▷ Near manifolds, the proximal gradient sends points to the manifold, smoothly:

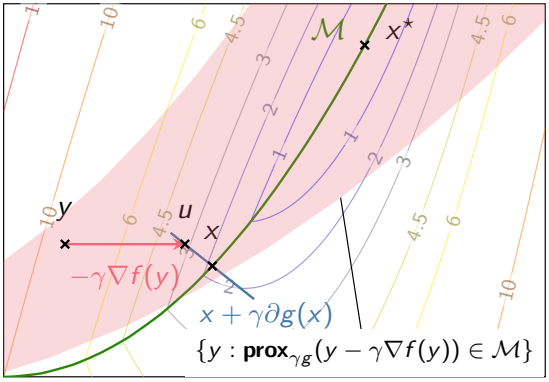
Theorem

Take y and $x = \mathbf{prox}_{\gamma g}(y - \gamma \nabla f(y))$ s.t.

- $x \in \mathcal{M}$ *partial smoothness*
- $y - \gamma \nabla f(y) - x \in \mathrm{ri}\ \gamma \partial g(x)$

then, on a neighborhood $\mathcal{N}_y$ of y the proximal gradient is

- ***$\mathcal{M}$-valued***
- *a $\mathcal{C}^1$ operator*



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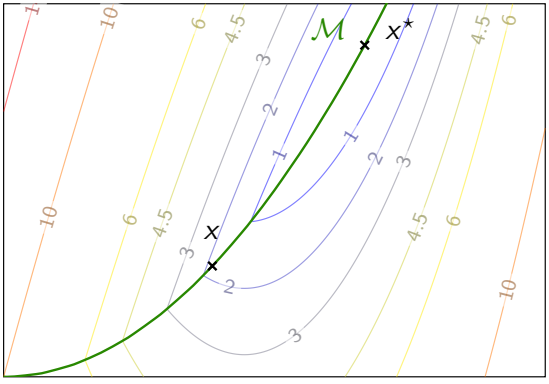
Riemannian optimization, in a nutshell



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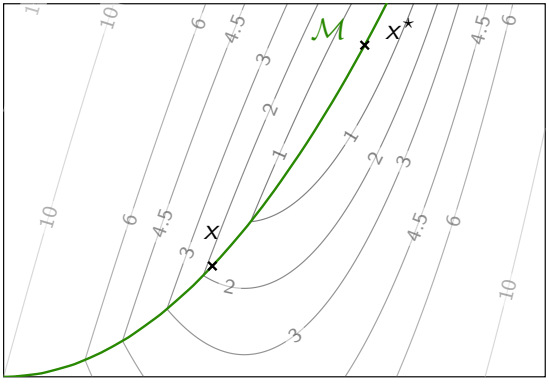

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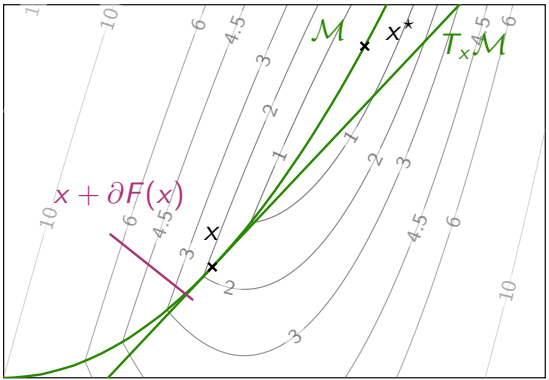

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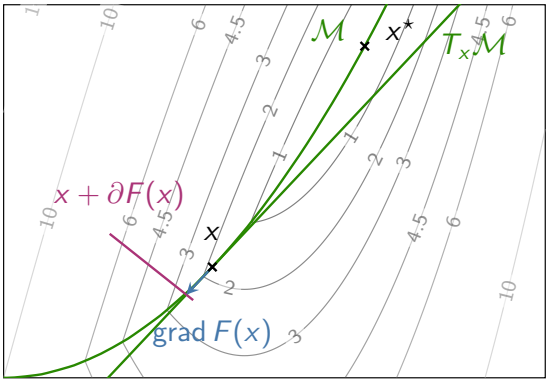

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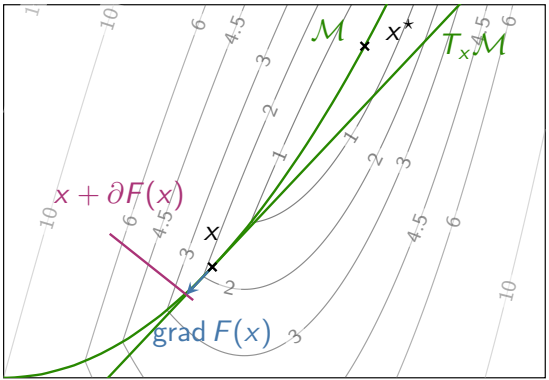

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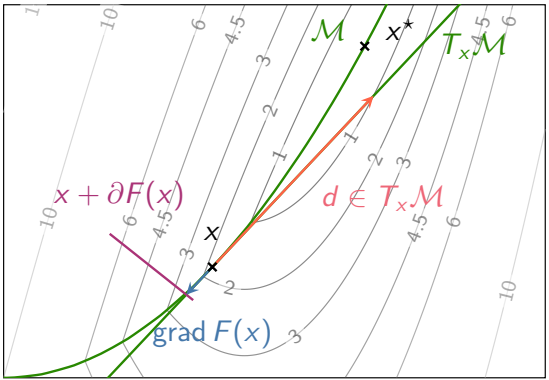

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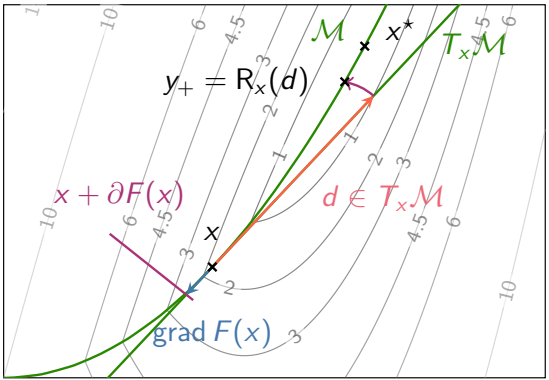

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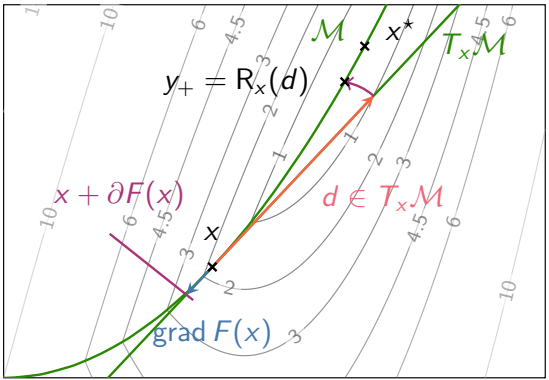
$\rightarrow$ Many methods of smooth optim carry over manifold optimization.

◇ Absil *et al* '09, Boumal '22



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Proposed algorithm

Iteration k:

- Compute *point* $x_k = \mathbf{prox}_{\gamma g}(y_{k-1} - \gamma \nabla f(y_{k-1}))$ and *manifold* $\mathcal{M}_k \ni x_k$
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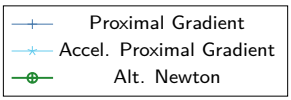
- *Assume that a Riemannian Newton method is used, and that one limit point $x^\star$ is such that*
  - *g has structure $\mathcal{M}^\star$ at $x^\star$*
  - *$0 \in \operatorname{ri} \partial F(x^\star)$, $\operatorname{Hess}_{\mathcal{M}^\star} F(x^\star) \succ 0$ and $\operatorname{Hess}_{\mathcal{M}^\star}$ is locally Lipschitz around $x^\star$*

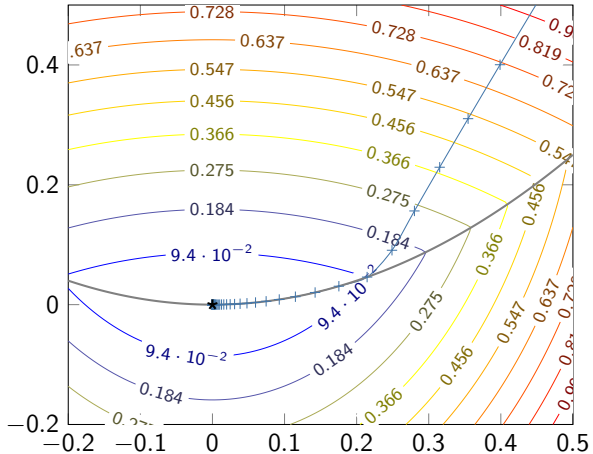
Then, after some finite time

- *$x_k \in \mathcal{M}^\star$*
- *x_k converges to $x^\star$ at a* ***quadratic rate****:* $\|x_{k+1} - x^\star\| \leq C\|x_k - x^\star\|^2$

Illustration: toy example

$$\min_{x \in \mathbb{R}^2} \underbrace{2x_1^2 + x_2^2}_{f(x)} + \underbrace{|x_1^2 - x_2|}_{g(x)}$$





◇ Beck Teboulle '09, Chambolle Dossal '15, B Iutzeler '20

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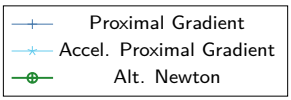

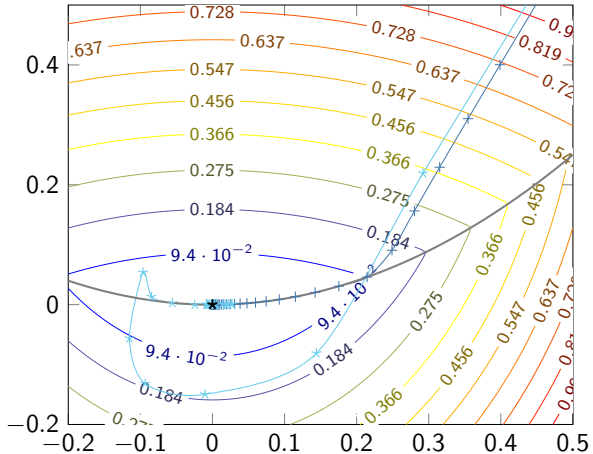

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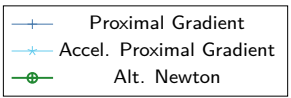

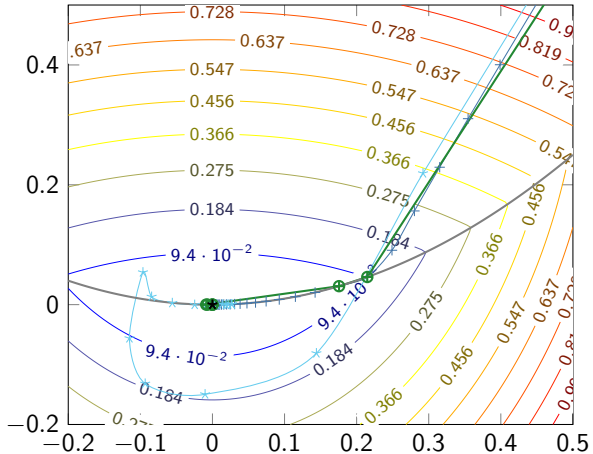

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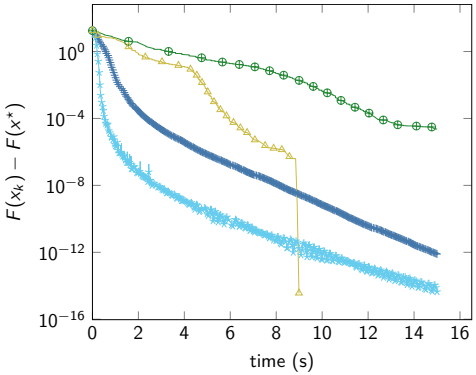
Illustration 1: logistic regression

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \log(y_i \sigma(\langle A_i, x \rangle)) + \lambda \|x\|_1$$

In this instance, $n = m = 4000$

$$\dim(\mathcal{M}^\star) = 249$$



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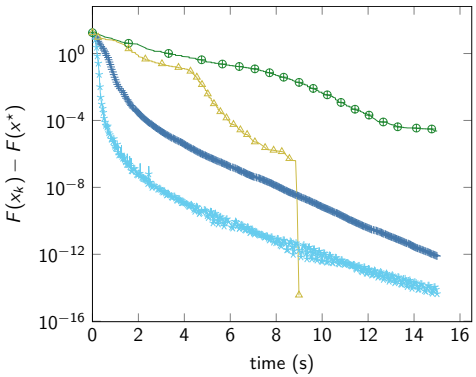
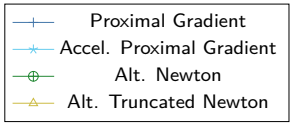




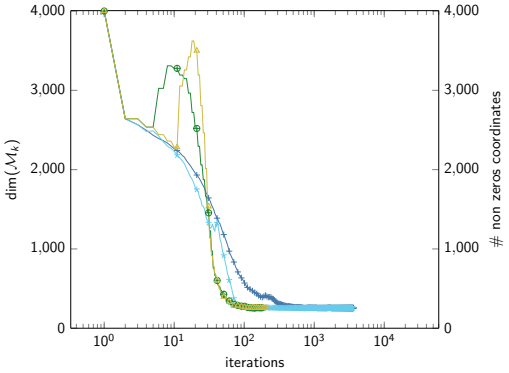
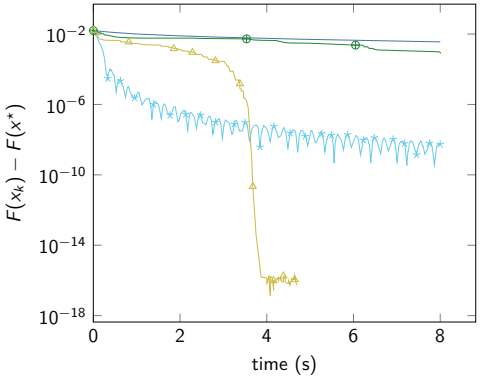


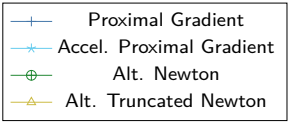
Illustration 2: tracenorm regression

$$\min_{X \in \mathbb{R}^{10\times 12}} \sum_{i=1}^{m} \left(\langle A_i, X\rangle - y_i\right)^2 + \lambda \|X\|_*$$

In this instance, $m = 60$

$$\mathcal{M}^\star = \{X : \operatorname{rank}(X) = 6\}$$





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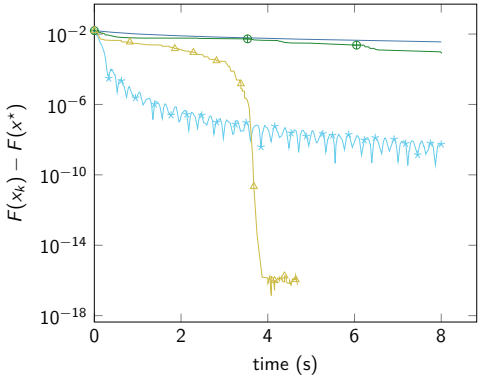

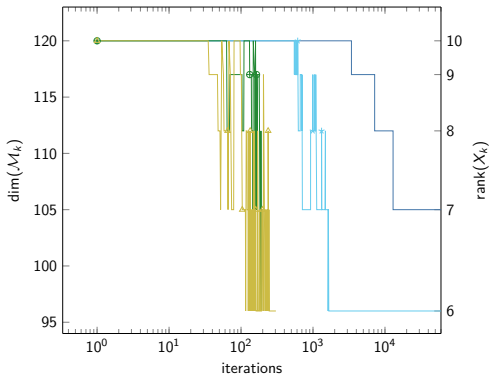

Summary on additive problems

Main messages:

- The proximal-gradient **identifies structure**
- With Riemannian Newton steps, we propose a global algorithm that
  - identifies $\mathcal{M}^\star$
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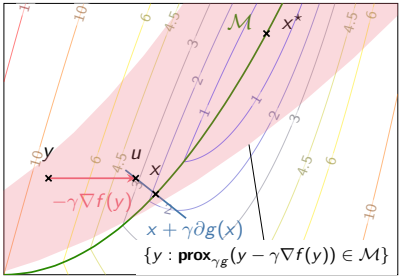

Proximal gradient identification

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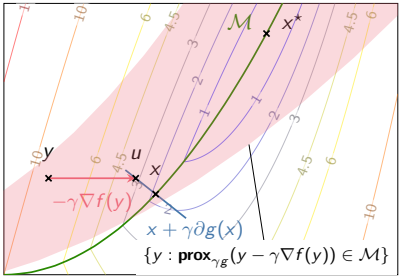
Extensions: This is the simplest scheme

- other Riemannian method cubic regularization of Newton
- other interleaving



Proximal gradient identification

Outline

Composite problem

$$\text{Find } x^\star \in \arg\min_{x \in \mathbb{R}^n} F(x) = g \circ c(x), \text{ with } g \text{ nonsmooth and } c \text{ a smooth mapping}$$

Example:

- Maximum of smooth functions: $F(x) = \max(c_1(x), \ldots, c_m(x))$ ◇ Womersley '86
- Spectral maximum: $F(x) = \lambda_{\max}(c(x))$, where $c(x) \in \mathbb{S}_m$ ◇ Oustry '00, Noll '05

Again, nonsmoothness is **explicit**: $\mathbf{prox}_{\gamma g}$ is available

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▷ Structure detection: iterates of optimization methods **do not belong to manifolds** anymore

- Composite bundle ◇ Sagastizábal '13
- proxlinear methods ◇ Lewis Wright '16

→ What structure near point x?

Back to the prox

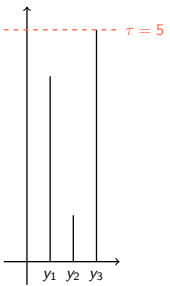
$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Example: Prox of max function

$$[\mathbf{prox}_{\gamma \max}(y)]_i = \begin{cases} \tau & \text{if } y_i \geq \tau \\ y_i & \text{else} \end{cases}$$

where τ solves $\sum_{\{i: y_i > \tau\}} (y_i - \tau) = \gamma$

$$\mathcal{M}_I = \{y : y_i = \max(y) \text{ for } i \in I\}$$



$y = (4, 1, 5)$

$\gamma = 0$

$\mathbf{prox}_{\gamma \max}(y) = (4, 1, 5)$

Back to the prox

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

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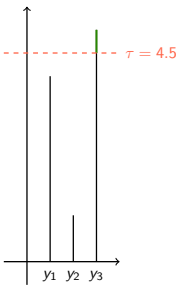

$y = (4, 1, 5)$

$\gamma = 0.5$

$\mathbf{prox}_{\gamma \max}(y) = (4, 1, 4.5)$

Back to the prox

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Example: Prox of max function

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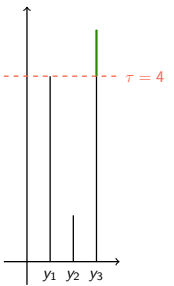

$y = (4, 1, 5)$

$\gamma = 1$

$\mathbf{prox}_{\gamma \max}(y) = (4, 1, 4)$

Back to the prox

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Example: Prox of max function

$$[\mathbf{prox}_{\gamma \max}(y)]_i = \begin{cases} \tau & \text{if } y_i \geq \tau \\ y_i & \text{else} \end{cases}$$

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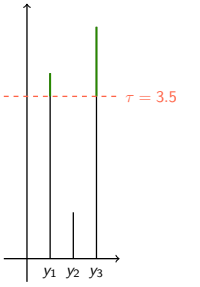

$y = (4, 1, 5)$

$\gamma = 2$

$\mathbf{prox}_{\gamma \max}(y) = (3.5, 1, 3.5)$

Back to the prox

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Example: Prox of max function

$$[\mathbf{prox}_{\gamma \max}(y)]_i = \begin{cases} \tau & \text{if } y_i \geq \tau \\ y_i & \text{else} \end{cases}$$

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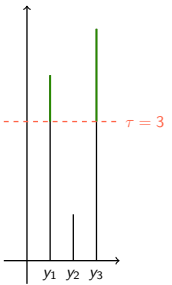
$$\mathcal{M}_I = \{y : y_i = \max(y) \text{ for } i \in I\}$$



$y = (4, 1, 5)$

$\gamma = 3$

$\mathbf{prox}_{\gamma \max}(y) = (3, 1, 3)$

Back to the prox

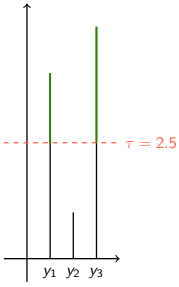
$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

Example: Prox of max function

$$[\mathbf{prox}_{\gamma \max}(y)]_i = \begin{cases} \tau & \text{if } y_i \geq \tau \\ y_i & \text{else} \end{cases}$$

where τ solves $\sum_{\{i:y_i > \tau\}} (y_i - \tau) = \gamma$

$$\mathcal{M}_I = \{y : y_i = \max(y) \text{ for } i \in I\}$$



$y = (4, 1, 5)$

$\gamma = 4$

$\mathbf{prox}_{\gamma \max}(y) = (2.5, 1, 2.5)$

Back to the prox

$$\mathbf{prox}_{\gamma g}(y) = \arg\min_u \left\{ g(u) + \frac{1}{2\gamma}\|u - y\|^2 \right\}$$

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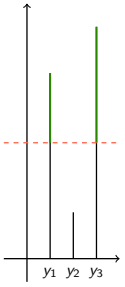


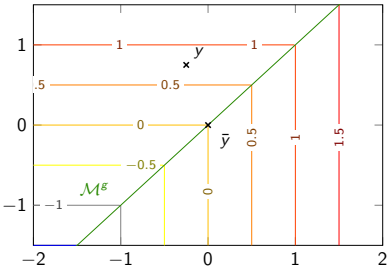
$y = (4, 1, 5)$

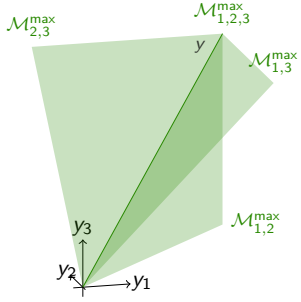
$\gamma = 4$

$\mathbf{prox}_{\gamma \max}(y) = (2.5, 1, 2.5)$

$\rightarrow$ Computing $\mathbf{prox}_{\gamma g}(y)$ also gives **structure information** $\mathcal{M} \ni \mathbf{prox}_{\gamma g}(y)$.

Identification for g, explicit $\mathrm{prox}_{\gamma g}$





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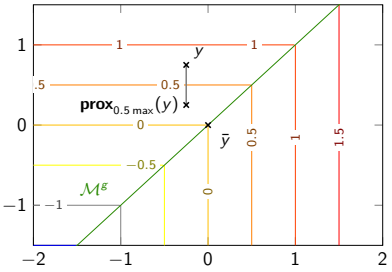

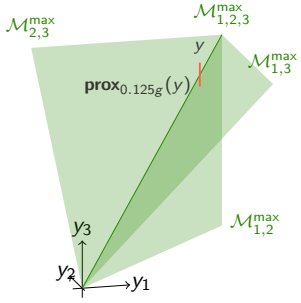

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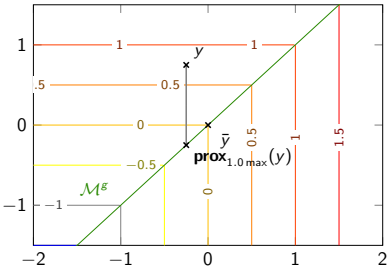

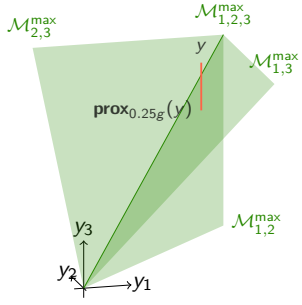

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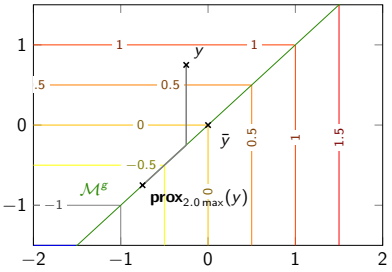

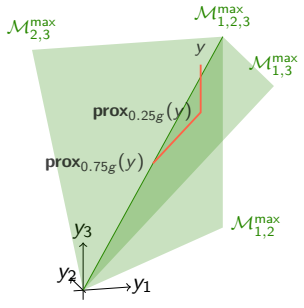

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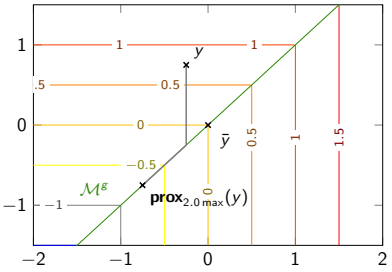
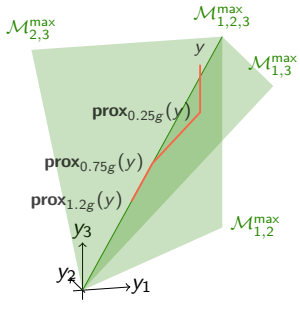




Identification for g, explicit $\mathbf{prox}_{\gamma g}$

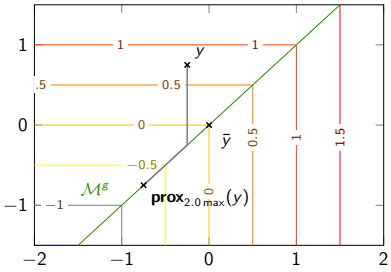
Lemma

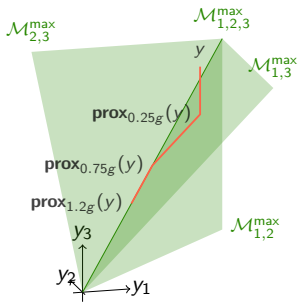
Consider a function g and point $\bar{y}$ with structure $\mathcal{M}^g$ that meet two technical assumptions. For all y near $\bar{y}$,

$$\mathbf{prox}_{\gamma g}(y) \in \mathcal{M}^g \qquad \text{for all } \gamma \in [\varphi^g(\mathrm{dist}_{\mathcal{M}^g}(y)), \Gamma^g]$$

where $\Gamma^g > 0$ and $\varphi^g(t) = \frac{1}{c_{ri}} t + \mathcal{O}(t^2)$.

Technical assumptions: normal ascent, control on projection curves





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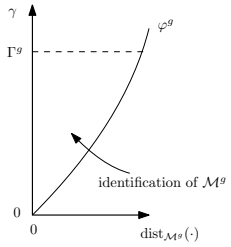
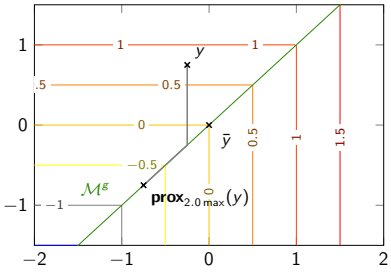
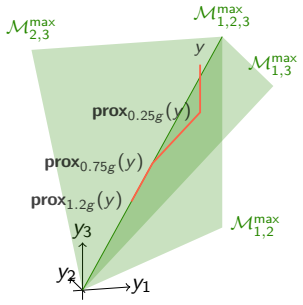
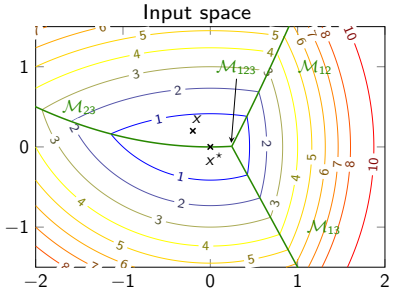




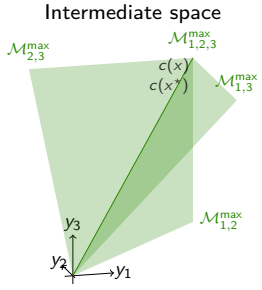


Identification for $g \circ c$, no $\text{prox}_{\gamma g \circ c}$

The prox of $F = g \circ c$ is **not available**, but **we do have** $\text{prox}_{\gamma g}$.



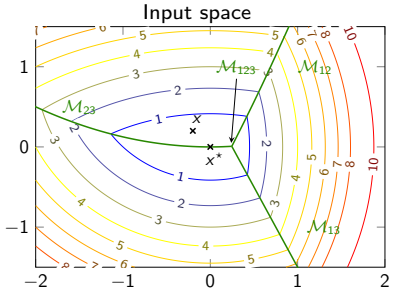
$F(x) = \max(c_1(x), c_2(x), c_3(x))$



$g(y) = \max(y_1, y_2, y_3)$

Identification for $g \circ c$, no $\mathbf{prox}_{\gamma g \circ c}$

The prox of $F = g \circ c$ is **not available**, but **we do have** $\mathbf{prox}_{\gamma g}$.



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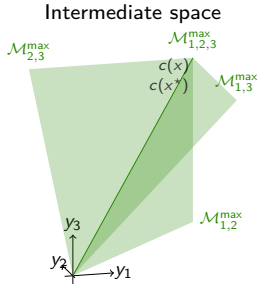


$g(y) = \max(y_1, y_2, y_3)$

Observation: $\mathbf{prox}_{\gamma g}$ can map points to $\mathcal{M}^g$.
The structure naturally lies in the intermediate space ... but that's ok!

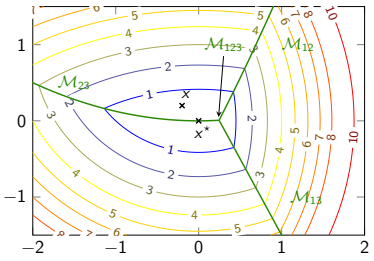
Identification for $g \circ c$

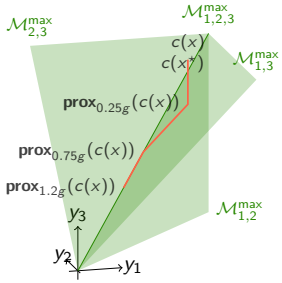
Theorem

Consider g, c smooth and a point $\bar{x}$ such that $c(\bar{x})$ has structure manifold $\mathcal{M}^g$ with c and $\mathcal{M}^g$ are transversal at $c(\bar{x})$. For all x near $\bar{x}$,

$$\mathbf{prox}_{\gamma g}(c(x)) \in \mathcal{M}^g \quad \text{for all } \gamma \in [\varphi(\mathrm{dist}_{\mathcal{M}}(x)), \Gamma]$$

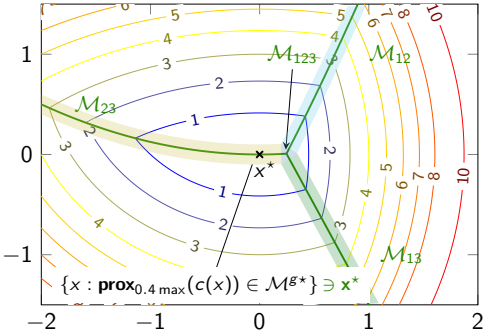
where $\Gamma > 0$ and $\varphi(t) = \frac{c_{map}}{c_{ri}} t + \mathcal{O}(t^2)$. Furthermore, $\mathcal{M} = c^{-1}(\mathcal{M}^g)$.





$[\varphi(\mathrm{dist}_{\mathcal{M}}(x)), \Gamma] \subset [0.25, 0.75]$

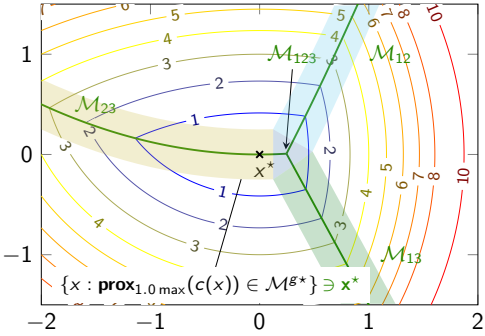
Towards an algorithm: i. detecting structure



γ too small $\Rightarrow$ detection of $\mathcal{M}^\star$ only near $x^\star$;

$$\mathcal{M}^\star = \mathcal{M}_{23}$$

Towards an algorithm: i. detecting structure



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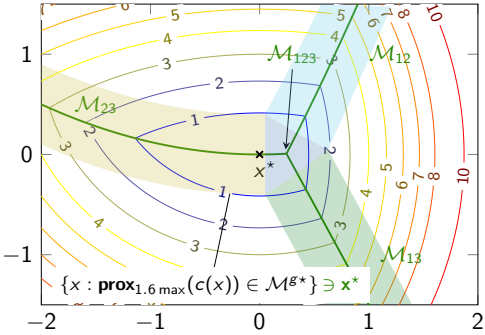

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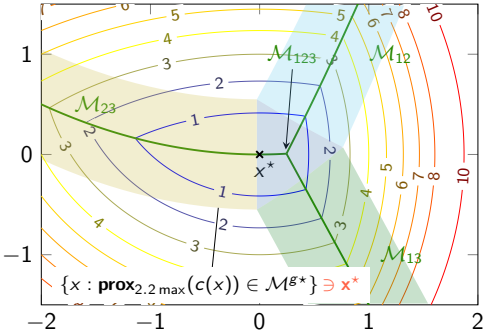

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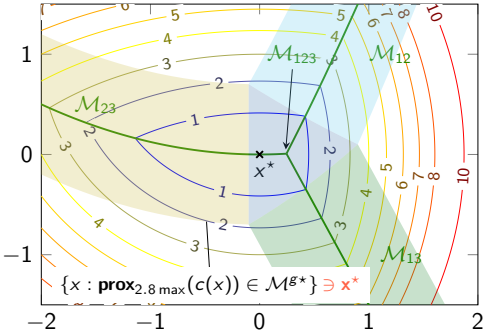

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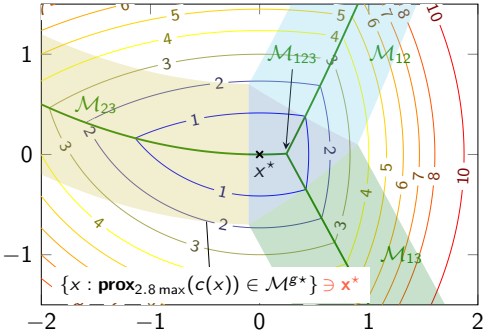
Towards an algorithm: i. detecting structure

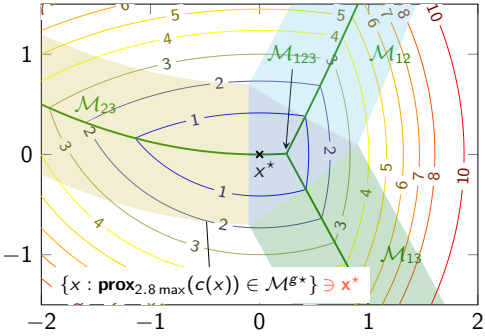


$\mathcal{M}^\star = \mathcal{M}_{23}$

γ too small $\Rightarrow$ detection of $\mathcal{M}^\star$ only near $x^\star$;

γ too large $\Rightarrow$ no detection of $\mathcal{M}^\star$ near $x^\star$.

Towards an algorithm: i. detecting structure



γ too small $\Rightarrow$ detection of $\mathcal{M}^\star$ only near $x^\star$;

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$$\mathcal{M}^\star = \mathcal{M}_{23}$$

Bottom line: $\mathbf{prox}_{\gamma g} \circ c(\cdot)$ detects $\mathcal{M}^\star$ from any x near $x^\star$ when $\gamma \in [\frac{c_{map}}{c_{ri}} \operatorname{dist}_{\mathcal{M}}(x), \Gamma]$.

$\rightarrow$ How to choose the step in practice?

Towards an algorithm: ii. exploiting structure

$$\begin{array}{lll} \mathcal{M} \text{ is smooth} & \exists h \text{ smooth s.t.} & x \in \mathcal{M} \Leftrightarrow h(x) = 0 \\ F \text{ smooth on } \mathcal{M} & \exists \widetilde{F} \text{ smooth s.t.} & F|_{\mathcal{M}} \equiv \widetilde{F} \text{ on } \mathcal{M} \end{array}$$

$$\min_{x \in \mathbb{R}^n} F(x) \quad \xrightarrow{\text{with } \mathcal{M}} \quad \min_{x} \widetilde{F}(x) \text{ s.t. } h(x) = 0.$$

Towards an algorithm: ii. exploiting structure

$$\begin{array}{lll} \mathcal{M} \text{ is smooth} & \exists h \text{ smooth s.t.} & x \in \mathcal{M} \Leftrightarrow h(x) = 0 \\ F \text{ smooth on } \mathcal{M} & \exists \widetilde{F} \text{ smooth s.t.} & F|_{\mathcal{M}} \equiv \widetilde{F} \text{ on } \mathcal{M} \end{array}$$

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▹ Many tools for smooth constrained optimization: Interior Point Methods, **Sequential Quadratic Programming**, Augmented Lagrangian Methods, *etc*

Algorithm: detection & exploitation

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- Compute $\mathbf{prox}_{\gamma_k g}(c(x_k))$ and obtain $\mathcal{M}_k$ and h_k, $\widetilde{F}_k$

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$$d_k^{\mathrm{SQP}}(x_k) = \underset{d \in \mathbb{R}^n}{\arg\min} \quad \langle \nabla \widetilde{F}_k(x_k), d \rangle + \frac{1}{2} \langle \nabla^2_{xx} L_k(x_k, \lambda_k(x_k)) d, d \rangle$$
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where $L_k(x, \lambda) = \widetilde{F}_k(x) + \langle \lambda, h_k(x) \rangle$, and $\lambda_k(x_k) = \arg\min_{\lambda \in \mathbb{R}^r} \left\| \nabla \widetilde{F}_k(x_k) + \sum_{i=1}^m \lambda_i \nabla h_{k,i}(x_k) \right\|^2$

Set $x_{k+1} = x_k + d_k^{\mathrm{SQP}}(x_k)$ if $F(x_k + d_k^{\mathrm{SQP}}(x_k)) < F(x_k)$

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▷ $\gamma_{k+1} = \frac{\gamma_k}{2}$

Local exact structure identification and quadratic convergence

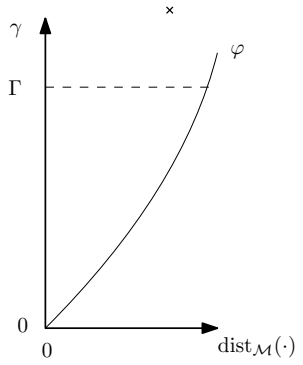
Theorem

Consider $F = g \circ c$ and a minimizer $x^\star \in \mathcal{M}^\star$ that meet some assumptions

Then, if x_0 is close enough to $x^\star$ and γ_0 is large enough, after some finite time

- $\mathcal{M}_k = \mathcal{M}^\star$
- x_k *converges to $x^\star$ at a* ***quadratic rate****:* $\|x_{k+1} - x^\star\| \leq C\|x_k - x^\star\|^2$

Sketch of proof, on a picture:



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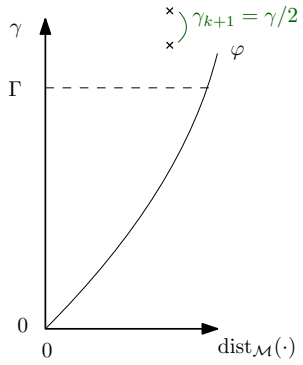

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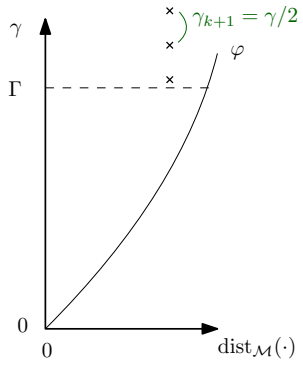

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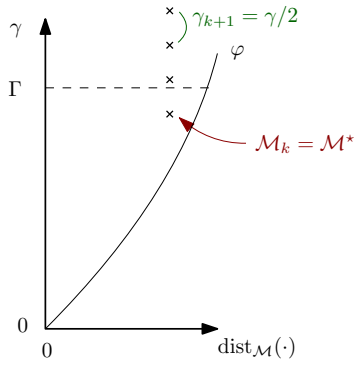

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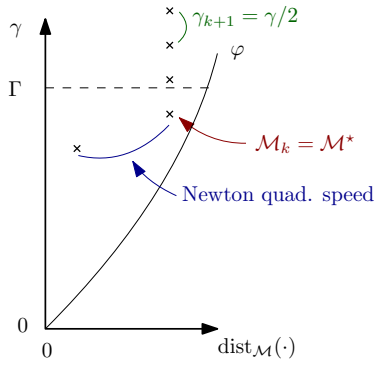

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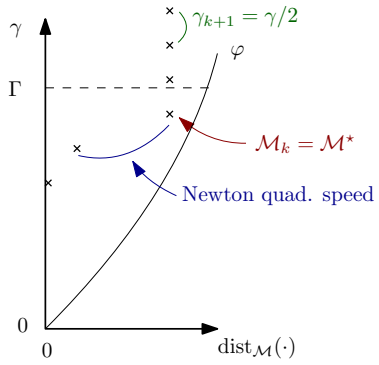

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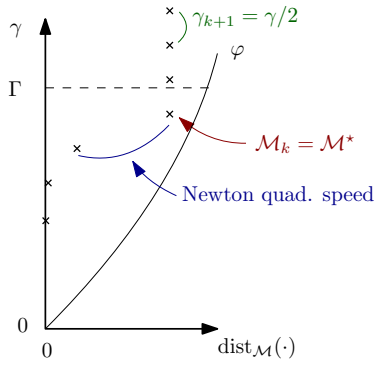
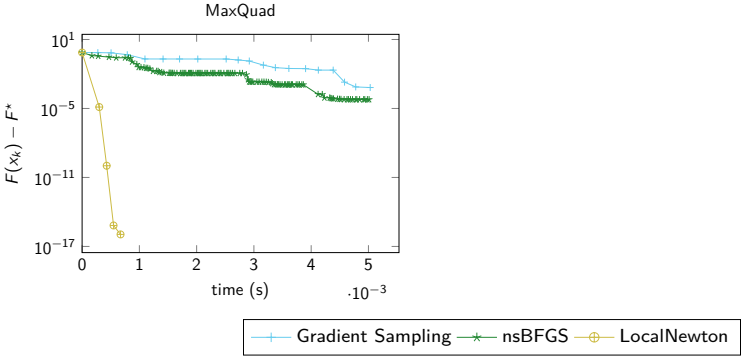


Illustrations: local behavior

$$\min_{x \in \mathbb{R}^{10}} \max_{i=1,\cdots,5} (c_i(x))$$

In this historical instance ◇ HULL '93

$$\mathcal{M}^\star = \{x : c_2(x) = \ldots = c_5(x)\}$$



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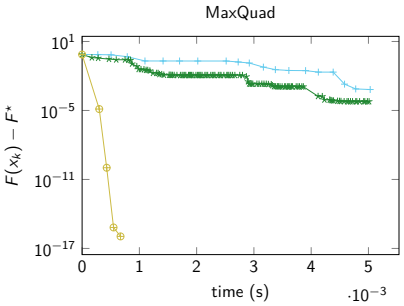
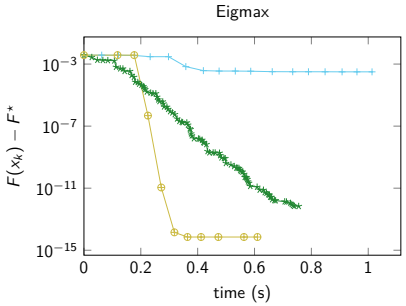


$$\min_{x \in \mathbb{R}^n} \lambda_{\max}\left(A_0 + \sum_{i=1}^{n} x_i A_i\right)$$

In this instance, $n = 25$, $A_i \in \mathbb{S}_{50}$

$$\mathcal{M}^\star = \{x : \lambda_{\max}(c(x)) \text{ has multiplicity } 3\}$$



Gradient Sampling — nsBFGS — LocalNewton

So far, on composite problems

Main messages:

- The proximal operator **identifies structure** in composite problems
- With Newton SQP steps, we propose a algorithm that **locally**
  - identifies $\mathcal{M}^\star$
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Do we really need to start near $x^\star$?

Towards a global nonsmooth Newton method

▷ **Difficulty # 1:** Newton's direction may fail to provide functional descent.

→ linesearch: find α such that $F(x + \alpha d) \leq F(x) + \alpha m F'(x; d)$.

- Is d a descent direction $F'(x; d) < 0$?
- The linesearch may jeopardizes the final quadratic rate? Maratos effect, *etc*

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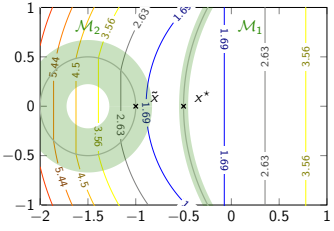
▷ **Difficulty # 2:** Detection of relevant structure, far from $x^\star$

Example In green, structure detection of $\mathbf{prox}_{\gamma g} \circ c$

Point $\tilde{x}$

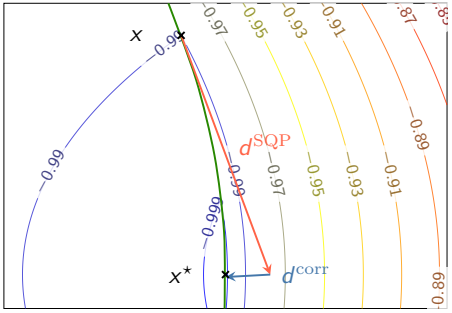
- is not a minimizer
- but stable structure detection by $\mathbf{prox}_{\gamma g} \circ c$.



No Maratos effect with correction

Maratos effect: $F(x + d^{\mathrm{SQP}}) > F(x)$
$\rightarrow$ Newton direction is not accepted

Second-order correction: from $x + d^{\mathrm{SQP}}$, go towards $\mathcal{M}$ Newton-Raphson step



Theorem (Local admissibility of unit stepsize)

Consider $F = g \circ c$ and a strong minimizer $x^\star \in \mathcal{M}^\star$ that meet some assumptions
Then if x is near $x^\star$ and $m \in (0, 1/2)$,

$$F\left(x + d^{\mathrm{SQP}} + d^{\mathrm{corr}}\right) \leq F(x) + mF'(x; d^{\mathrm{SQP}}).$$

Heuristic algorithm illustrated

$$\min_{x \in \mathbb{R}^n} \lambda_{\max}\left(A_0 + \sum_{i=1}^{n} x_i A_i\right)$$

In this instance, $n = 25$, $A_i \in \mathbb{S}_{50}$

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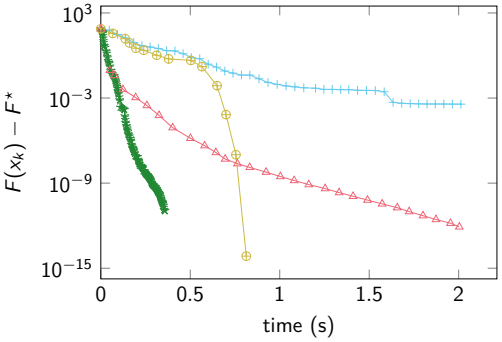

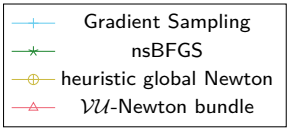

(our implementation of ◇ Mifflin Sagastizábal '05)

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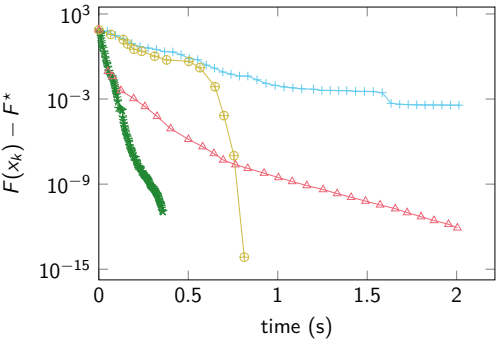

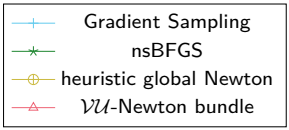

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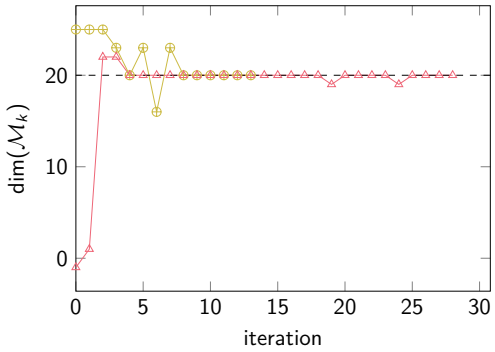
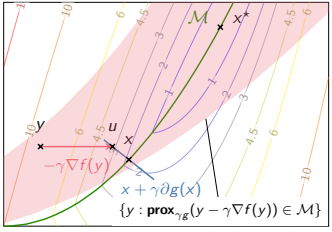


Outline

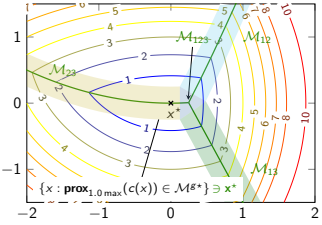
Three years, Three years, in one slide

We have made some steps towards

detecting and **exploiting** structure in nonsmooth optimization



Proximal gradient identification



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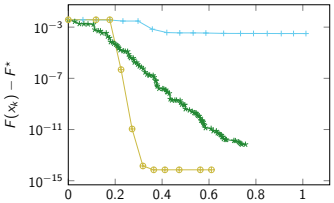

Quadratic convergence
on nonsmooth functions

Other aspects

Reproducibility

All experiments have online open source implementation in Julia

`gbareilles.fr/software`

Not discussed but important & interesting

- *active-set* solver on spectraplex
- implementation of $\mathcal{VU}$-bundle algorithm
- second derivatives of $\|\cdot\|_*$ and $\lambda_{\max}$ near nonsmooth points!
- nonconvex setting

Perspectives

Extensions

- Globalize local Newton method for $g \circ c$ – work in progress
- Smarter second-order updates: large-scale settings, efficient even away from minimizers
  $\rightarrow$ Investigate quasi-Newton (BFGS), cubic regularization of Newton
- Benchmarking: further numerical comparison $\diamond$ Massias *et al* '22

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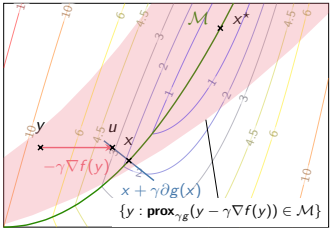
Applications

- Richer additive structures $f + g + h(L\cdot)$, write as $g \circ c$ with g prox-simple
  $\rightarrow$ Apply Local Newton Algorithm
- Smooth minimization on structured sets: $\min_{x \in \mathcal{S}} f(x) \rightarrow \min_x f(x) + \imath_{\mathcal{S}}(x)$
  $\rightarrow$ Newton acceleration of projected gradient

Thank you for your *presence* and *attention*!



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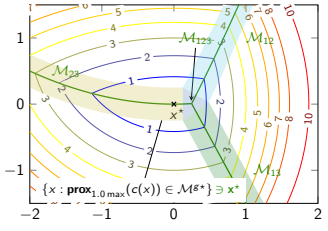

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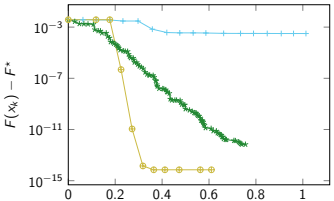

Quadratic rates
on nonsmooth functions